Timing Recovery for Synchronous Binary Data Transmission

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This paper analyzes different methods of adjusting the sampling time for detecting synchronous binary data, based on properties of the random data signal itself. The static error and the variance of the jitter of the resultant sampling instant are calculated where the effects of frequency offset, additive noise, signal overlap, and jitter of the reference source are included.

The threshold crossing timing recovery system adjusts the sampling time in response to the times at which the data signal crosses the amplitude threshold. The sampled-derivative system uses the time derivative of the signal at the sampling time to adjust sampling phase. It is shown that both systems lead to approximately the same amount of jitter in the presence of noise and signal overlap for a given bandwidth of the control loop.

An improved timing recovery system is presented which is constructed by adding correction signals to the sampled-derivative system. This system accounts for intersymbol interference in a manner that tends to set the sampling time at the point of maximum eye opening, where the error probability is minimum for the most adverse message sequence.

I. INTRODUCTION AND SUMMARY OF RESULTS

In synchronous polar binary data transmission, information is sent by serially transmitting either a basic signaling waveform or its negative at fixed time intervals. Modulation may be used to better fit the signal to the channel. At the receiving end, the signal is demodulated and filtered. The resultant baseband signal is sampled periodically, and the polarities at the sampling instants determine the output data. The choice of sampling time is critical for minimizing the error probability due to intersymbol interference and noise, particularly when the signal has been subjected to sharp cutoff filtering. The sampling time is best set by using some properties of the data signal itself.

The problem of timing is particularly acute in pulse code modulation (PCM) systems, where the accumulation of jitter in a long chain of regenerators frequently limits the allowable length of such systems. For this reason, previous studies of timing recovery have concentrated on PCM applications.[1-4] The use of a tuned circuit as the memory element is generally assumed, since this is commonly employed in PCM repeaters.

This paper will concentrate on timing recovery for data transmission applications. The effects of multiple regeneration will not be considered. The recovery of timing will be accomplished by a feedback control system, such as a phase-locked loop. Different methods of generating the error signal for the control loop will be compared.

The received signal, after demodulation and filtering, is of the form

$$s(t) = \sum_{k=-\infty}^{\infty} a_k f(t - \beta T - kT) + n(t) \tag{1}$$

where $\{a_k\}$ is a set of independent random variables, each equal to $+1$ or -1 with equal probability. This may be assured by the use of a scrambler if the data source is itself not random. The basic signaling waveform is $f(t)$. The abscissa of $f(t)$ will be adjusted for each system to be studied so that the desired sampling time of $f(t)$ is $t = 0$. The quantity β is an unknown fractional time delay. Since we are not concerned with absolute time delay between transmitter and receiver, we will assume $| \beta | \leqq \frac{1}{2}$. The additive noise is $n(t)$.

The sampling wave which determines the times at which $s(t)$ is sampled may be represented by

$$q(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT - \gamma T), \tag{2}$$

where γ is a phase that is generally time varying.

The output data is determined by

$$\hat{a}_n = \operatorname{sgn} s(nT + \gamma T), \tag{3}$$

where $\operatorname{sgn} v = v/| v |$. Then

$$\hat{a}_n = \operatorname{sgn} [a_n f(\gamma T - \beta T) + \sum_{k \neq 0} a_{n-k} f(kT + \gamma T - \beta T) + n(t)]. \tag{4}$$

For simplicity, the argument of the noise term is not made explicit since it is of no consequence. Assuming that $f(\gamma T - \beta T)$ is positive, then $\hat{a}_n$ will agree with a_n provided that

$$-a_n[\sum_{k \neq 0} a_{n-k} f(kT + \gamma T - \beta T) + n(t)] < f(\gamma T - \beta T). \tag{5}$$

It is readily seen that the probability that (5) holds depends strongly on $\gamma - \beta$. For each timing recovery system, $f(t)$ is defined so that the desired value of $\gamma - \beta$ is zero.

The principal part of a timing recovery system is a phase detector which examines $s(t)$ and $q(t)$ and attempts to generate an error signal proportional to $\beta - \gamma$. It is not possible to determine β exactly from $s(t)$ since the signaling waveform and the noise are unknown. This paper will consider different methods of forming this estimate of $\beta - \gamma$.

Another essential component of the system is the reference source which is used to generate the sampling wave. Its phase or frequency is adjusted by the error signal in order to form a sampling wave of the proper phase. The error signal may be filtered prior to its use in shifting the phase of the reference source. The reference source may be a local oscillator whose natural frequency is set as close as possible to the bit rate. The reference source may instead be derived from transmitted pilot tones, in which case its frequency is exact, but it might have phase jitter of its own due to channel noise.

Section II describes and analyzes a timing recovery system which uses a threshold crossing phase detector. This detector generates an error signal each time the signal crosses zero. The amplitude of this signal is proportional to the difference between the time of occurrence of the zero crossing and the time of the nearest sampling pulse, displaced by half a bit period. This system tends to choose a sampling instant which is midway between the mean transition times.

The sampled-derivative phase detector is discussed in Section III. This device generates an error signal during each bit interval which is proportional to the time derivative of the signal at the sampling time multiplied by the signal polarity at that time. The sampled-derivative timing recovery system attempts to set the sampling time to coincide with the peak of $f(t)$.

The analysis shows that the performance of these two systems is very similar for a given open loop gain function of the control system, $G(\omega)$. Approximations are made based on the assumption that the phase error is small and that $G(\omega)$ is a narrowband low-pass function compared with the bit rate.

The systems fail if $G(0)$ is finite and the reference source does not agree exactly in natural frequency with the bit rate. If the reference source has the correct frequency and a phase θ, then a static phase error

$$\bar{e} = \frac{\theta - \beta}{1 + G(0)} \tag{6}$$

results in the sampling wave.

Much better performance can be achieved if $G(\omega)$ has a pole at the origin. In this case the system is insensitive to the phase of the reference source. In the presence of a frequency error, Δf, the static error is

$$\bar{e} = -j\,\Delta f \frac{d}{d\omega}\left[\frac{1}{1 + G(\omega)}\right]_{\omega=0}. \tag{7}$$

In addition to any static error, the sampling time will also jitter about its mean value. The variance of this phase jitter is the sum of several components, each due to a different cause. Jitter is produced by jitter in the reference source, by additive noise and by signal overlap. In the case of the threshold crossing system, jitter is also introduced whenever there is a static error.

If the reference source has a jitter whose power spectral density is $S_\eta(\omega)$, then the output fractional jitter will have a variance equal to

$$\sigma_{RS}^2 = \frac{1}{\pi}\int_0^\infty \frac{S_\eta(\omega)}{|1 + G(\omega)|^2}\,d\omega. \tag{8}$$

This indicates that high-frequency noise components must be removed from the reference source prior to its use for timing recovery.

The jitter produced by the additive noise is

$$\sigma_N^2 = \frac{2[R_n(0) - R_n(T)]}{T[f'(-T/2) - f'(T/2)]^2}\,\omega_1 \tag{9}$$

for the threshold crossing system. $R_n(t)$ is the autocorrelation of the noise and ω_1 is the noise bandwidth of the closed control loop.

$$\omega_1 = \frac{1}{\pi}\int_0^\infty \left|\frac{G(\omega)}{1 + G(\omega)}\right|^2 d\omega. \tag{10}$$

For the sampled-derivative system, the noise leads to jitter variance

$$\sigma_N^2 = -\frac{R_n''(0)}{Tf''^2(0)}\,\omega_1\,. \tag{11}$$

In typical data transmission systems, (9) and (11) are similar in magnitude, and not very sensitive to the shape of $f(t)$ if the noise is similarly filtered.

The jitter variance due to signal overlap is of the form

$$\sigma_S^2 = A_1\omega_1 T + A_2(\omega_2 T)^3, \tag{12}$$

where

$$\omega_2^3 = \frac{1}{\pi}\int_0^\infty \omega^2 \left|\frac{G(\omega)}{1 + G(\omega)}\right|^2 d\omega. \tag{13}$$

If $\omega_1 T$ and $\omega_2 T$ are comparable and much less than unity, then the first term is usually much larger than tbe second.

For the threshold crossing system,

$$A_1 = \frac{1}{T^2[f'(-T/2) - f'(T/2)]^2} \sum_{k=-\infty}^{\infty} [f(kT + T/2) - f(kT - T/2)] \cdot [2f(kT + T/2) + f(-kT + T/2) - f(-kT - T/2)] \tag{14}$$

and

$$A_2 = \frac{1}{2T^2[f'(-T/2) - f'(T/2)]^2} \Big\{ -4f^2(T/2) + 2 \sum_{k=-\infty}^{\infty} f(kT + T/2)f(kT - T/2) - \sum_{k=-\infty}^{\infty} k^2[f(kT + T/2) - f(kT - T/2)][f(-kT + T/2) - f(-kT - T/2)] \Big\}. \tag{15}$$

For the sampled-derivative system,

$$A_1 = \frac{1}{T^2 f''^2(0)} \sum_{k=-\infty}^{\infty} f'(kT)[f'(kT) + f'(-kT)] \tag{16}$$

and

$$A_2 = \frac{-1}{2T^2 f''^2(0)} \sum_{k=-\infty}^{\infty} k^2 f'(kT) f'(-kT). \tag{17}$$

In botb cases, $A_1 = 0$ if $f(t)$ is an even function, so tbe timing recovery systems are very sensitive to asymmetry of the basic signaling waveform. The jitter variances are again comparable for both systems. As may be expected, the jitter increases considerably as the filter used to shape $f(t)$ is made sharper.

There is an additional jitter component for the threshold crossing system whenever tbere is a static error. Its variance is given by

$$\sigma_e^2 = \bar{e}^2 \omega_1 T. \tag{18}$$

An example is provided in Section IV. A typical data transmission system using a distorted signal is studied so as to illustrate the magnitudes of the above quantities and to indicate the narrowness of loop bandwidth required for satisfactory performance.

In this example it is also seen that neither the threshold crossing timing recovery system nor the sampled-derivative system chooses a mean sampling time which is very near to the time at which the eye pattern has its maximum opening.

Section V describes an improved timing recovery system whose mean sampling time coincides with that of the maximum eye opening. This system is constructed by adding correction signals to the sampled-derivative system in order to account for the effects of intersymbol interference on the mean sampling time.

Finally, an outline of some extensions and modifications of these timing recovery systems is presented in Section VI.

II. THE THRESHOLD CROSSING SYSTEM

Most timing recovery systems make use of the instants at which the data signal crosses the threshold to alter the phase of the sampling wave. A block diagram of a typical threshold crossing timing recovery system is shown in Fig. 1.

The principal part in this system is the threshold crossing detector. This device generates an error pulse each time the signal crosses zero. The amplitude of the error pulse is proportional to the difference between the time of occurrence of the threshold crossing and the time of the nearest pulse of the displaced sampling wave. The displaced sampling wave is

$$q_d(t) = q(t - T/2) = \sum_{n=-\infty}^{\infty} \delta(t - nT - T/2 - \gamma T), \tag{19}$$

where γ is the phase measured in fractional signal periods. If the axis crossing following the mth sampling time is displaced by $\alpha_m T$,

$$s(mT + T/2 + \alpha_m T) = 0, \qquad |\alpha_m + \gamma| < \tfrac{1}{2}, \tag{20}$$

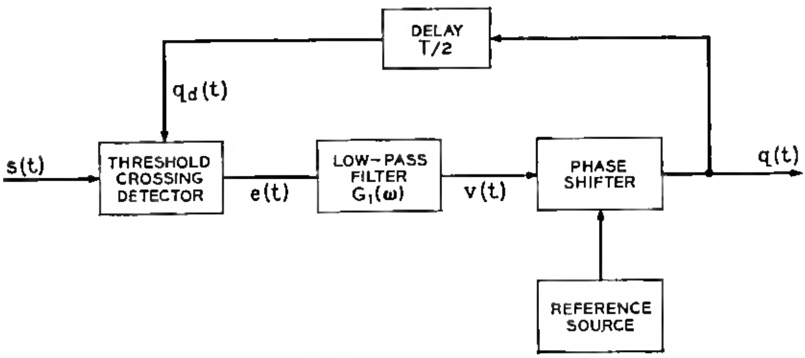

Fig. 1 — Threshold crossing timing recovery system.

then the error signal is

$$e(t) = \sum_m K_1(\alpha_m - \gamma)\, \delta(t - mT), \tag{21}$$

where K_1 is a gain constant. The error pulse has heen represented in (21) as an ideal impulse function. Since the error signal will he passed through a narrow low-pass filter, the response will he virtually identical to that when a more realistic pulse of the same area is used. Similarly, the effects of variation of the position of the pulse within the interval may he neglected, since the low-frequency components of the error signal are suhstantially uneffected.

We will now determine the threshold crossing time α_m as a function of the signal overlap and noise. Suhstituting (1) into (20) yields

$$\sum_{k=-\infty}^{\infty} a_{m-k} f(kT + T/2 - \beta T + \alpha_m T) + n(t) = 0. \tag{22}$$

If

$$f(T/2) + f(-T/2) > \sum_{k \neq 0,-1} |\, f(kT + T/2) \,| + n(t) \tag{23}$$

then a crossing will occur following the mth hit, if and only if $a_m = -a_{m+1}$. When $a_m = -a_{m+1}$ and (23) holds, (22) may he written as

$$a_m[f(T/2 - \beta T + \alpha_m T) - f(-T/2 - \beta T + \alpha_m T)] + \sum_{k \neq 0,-1} a_{m-k} f(kT + T/2 - \beta T + \alpha_m T) + n(t) = 0. \tag{24}$$

If $\alpha_m - \beta$ is small, we may approximate (24) hy the first terms of its Taylor series expansion.

$$a_m[f(T/2) - f(-T/2)] + a_m(\alpha_m - \beta)T[f'(T/2) - f'(-T/2)] + \sum_{k \neq 0,-1} a_{m-k} f(kT + T/2) + n(t) \approx 0. \tag{25}$$

Let the ahscissa of the function $f(t)$ be adjusted so that

$$f(T/2) = f(-T/2). \tag{26}$$

Then define

$$b = f'(-T/2) - f'(T/2). \tag{27}$$

We may now solve (25) for α_m.

$$\alpha_m \approx \beta + \frac{a_m}{bT} \Big[\sum_{k \neq 0,-1} a_{m-k} f(kT + T/2) + n(t) \Big]. \tag{28}$$

If we let

$$d_n = \begin{cases} 0, & \text{if} \quad a_n = a_{n+1} \\ 1, & \text{if} \quad a_n = -a_{n+1} \end{cases} \tag{29}$$

then the error signal (21) becomes

$$e(t) = K_1 \sum_{n=-\infty}^{\infty} d_n \Big\{ \beta - \gamma + \frac{a_n}{bT} \Big[\sum_{k \neq 0, -1} a_{n-k} f(kT + T/2) + n(t) \Big] \Big\} \delta(t - nT). \tag{30}$$

The error signal is passed through the filter $G_1(\omega)$ and then shifts the phase of a reference source. The reference source may be a local oscillator whose frequency is tuned as closely as possible to the signaling rate. Alternatively, the reference source may be derived from pilot tones which are transmitted along with the data. In the latter case, there is no error in the average frequency of the reference source, but its phase may be poorly related to that of the data signal and may also be perturbed by noise. In either case, the reference source generates a signal of the form

$$r(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT - \tau T). \tag{31}$$

When a local oscillator is used,

$$\tau \approx \Delta f t + \theta, \qquad \Delta f T \ll 1 \tag{32}$$

where Δf is the frequency offset of the oscillator and θ is an arbitrary constant. When the reference source is derived from pilot tones,

$$\tau = \theta + \eta(t) \tag{33}$$

where $\eta(t)$ is a zero mean random variable.

The sampling wave is formed by shifting the phase of the reference source by an amount proportional to the value of the filtered error signal.

$$q(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT - \tau T - K_2 v T), \tag{34}$$

where v is the filtered version of $e(t)$ and K_2 is the proportionality constant. Comparing (34) with (2),

$$\gamma = \tau + K_2 v = \tau + K_2 \int g_1(t - x) e(x) \, dx. \tag{35}$$

In the phase-locked loop type of control system, the frequency of the reference source is adjusted rather than its phase. This, however, is completely equivalent to integrating the error signal prior to phase adjustment. The block diagram is therefore valid for the phase-locked loop provided that the filter includes a pole at the origin. As will be shown, this pole is highly desirable and, in some cases, absolutely essential.

Substituting (35) into (30)

$$e(t) = K_1 \sum_{n=-\infty}^{\infty} d_n \Big\{ \beta - \tau - K_2 \int g_1(t-x)e(x)\,dx + \frac{a_n}{bT}\Big[\sum_{k\neq 0,-1} a_{n-k}f(kT+T/2) + n(t)\Big]\Big\}\,\delta(t-nT). \tag{36}$$

Let

$$e_2(t) = \frac{2}{K_1}\,e(t) \tag{37}$$

$$g_2(t) = \frac{K_1K_2}{2}\,g_1(t) \tag{38}$$

$$e(t) = \sum_{n=-\infty}^{\infty} e(n)\ \delta(t-nT) \tag{39}$$

and normalize the time variable so tbat $T = 1$. Then (36) can be written as

$$e_2(n) = 2d_n\Big\{\beta - \tau - \sum_{k=-\infty}^{n} g_2(n-k)e_2(k) + \frac{a_n}{b}\Big[\sum_{k\neq 0,-1} a_{n-k}f(k+1/2) + n(t)\Big]\Big\}. \tag{40}$$

A model of the threshold crossing timing recovery system which conforms with (40) is shown in Fig. 2. This is not a time-invariant linear system hecause of the presence of the multiplier.

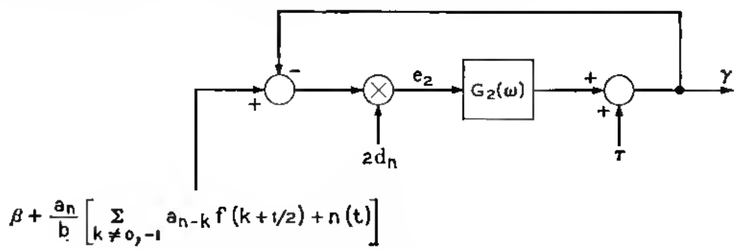

Fig. 2—Model of threshold crossing system.

In Appendix A, it is shown that

$$\overline{e_2(n)} = \beta - \bar{\tau} - g_2(0)\overline{e_2(n)} - \sum_{k=-\infty}^{n} g_2(n-k)\overline{e_2(k)}. \tag{41}$$

This system may be readily analyzed either by means of the z-transform, or equivalently, by discrete Fourier analysis.[2] The discrete Fourier transform is given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) \exp(-j\omega n). \tag{42}$$

If $G_2(\omega)$ is bandlimited to $|\omega| < \pi$, which is approximately true in all cases of interest, then these transforms will coincide with the true Fourier transforms.

The solution of (41) in terms of Fourier transforms is

$$\overline{E_2}(\omega) = \frac{\beta(\omega) - \bar{\tau}(\omega)}{1 + g_2(0) + G_2(\omega)}, \tag{43}$$

where

$$g_2(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_2(\omega)\, d\omega. \tag{44}$$

The static error can now be found from

$$\bar{\gamma}(\omega) - \beta(\omega) = G_2(\omega)E_2(\omega) + \bar{\tau}(\omega) - \beta(\omega) \tag{45}$$

$$\bar{\gamma}(\omega) - \beta(\omega) = \frac{1 + g_2(0)}{1 + g_2(0) + G_2(\omega)}\,[\bar{\tau}(\omega) - \beta(\omega)]. \tag{46}$$

In particular, if $\beta(t)$ is a constant, β_0, and $\tau(t)$ is given by (32), then

$$\bar{\gamma}(\omega) - \beta_0\,\delta(\omega) = 2\pi\frac{1 + g_2(0)}{1 + g_2(0) + G_2(\omega)}\,[j\,\Delta f\,\delta'(\omega) + (\theta - \beta_0)\delta(\omega)] \tag{47}$$

$$\bar{\gamma}(t) - \beta_0 = \frac{1 + g_2(0)}{1 + g_2(0) + G_2(0)}\,[\Delta f t + \theta - \beta_0] - j\,\Delta f\frac{d}{d\omega}\left[\frac{1 + g_2(0)}{1 + g_2(0) + G_2(\omega)}\right]_{\omega=0}. \tag{48}$$

If $G_2(0)$ is finite, then the first term is a steadily increasing error and the system fails. If $\Delta f = 0$, the system does not fail, but a static error will be present due to the arbitrary phase values, β_0 and θ. It is, therefore, highly desirable that $G(\omega)$ have a pole at the origin. In the presence of frequency offset, this pole is essential. In this case, only

the last term in (48) is nonzero, so that a frequency offset leads to a static error. The system is completely insensitive to the values of the arbitrary phases, β_0 and θ, except during initial start-up.

For the system to be stable, it is required that $1 + g_2(0) + G_2(\omega)$ have no zeros in the half plane $Im(\omega) < 0$. In most cases, $g_2(0) \ll 1$, so that the usual criteria for stability apply to $G_2(\omega)$.

We now wish to calculate the variance of γ, or the mean square jitter. From (40) and (41), and assuming the static error to be constant,

$$e_3(n) = e_2(n) - \overline{e_2} = 2d_n\Bigg[\overline{e_2} + g_2(0)\overline{e_2} - \sum_{k=-\infty}^{n} g_2(n-k)e_3(k) - \eta(t) + \frac{a_n}{b}n(t)\Bigg] + z(n) - \overline{e_2}\,, \tag{49}$$

where

$$z(n) = \frac{2}{b}\,d_n a_n \sum_{k\neq 0,-1} a_{n-k}f(k + 1/2). \tag{50}$$

Let

$$x(n) = 2d_n\Bigg[\overline{e_2} + g_2(0)\overline{e_2} + \frac{a_n}{b}n(t)\Bigg] + z(n) - \overline{e_2}\,. \tag{51}$$

Then (49) may be written as

$$e_3(n) = x(n) - 2d_n\eta(t) - 2d_n\sum_{k=-\infty}^{n} g_2(n-k)e_3(k). \tag{52}$$

The zero mean component of the output phase error is

$$\gamma_1(n) = \gamma(n) - \overline{\gamma} = \eta(t) + \sum_{k=-\infty}^{n} g_2(n-k)e_3(k) \tag{53}$$

$$\gamma_1(n) = \eta(t) + \sum_{k=-\infty}^{n} g_2(n-k)[x(k) - 2d_k\gamma_1(k)]. \tag{54}$$

The autocorrelation of (54) is

$$\sum_k\sum_l E\{[2d_k g(n-k) + \delta_{nk}][2d_l g(n+m-l) + \delta_{n+m,l}]\gamma_k\gamma_l\}$$
$$= R_\eta(m) + \sum_k\sum_l g_2(n-k)g_2(n+m-l)R_x(k-l), \tag{55}$$

where $E(v)$ denotes the expected value of v and $R_v(m) = E[v(n)v(n+m)]$ is the autocorrelation of v.

In almost all cases of practical interest, $g_2(0) \ll 1$. Then we may approximate

$$\bar{x} = g_2(0)\overline{e_2} \approx 0 \tag{56}$$

$$\overline{d_n\gamma(n)} = \tfrac{1}{2}\bar{x} \approx 0 \tag{57}$$

and

$$\overline{e_2} = \frac{\beta_0 - \overline{\gamma}}{1 + g_2(0)} \approx \beta_0 - \overline{\gamma}. \tag{58}$$

Subject to these approximations, we can evaluate the discrete Fourier transform of (55). After some algebraic manipulation, the result obtained is

$$| 1 + G_2(\omega) |^2 S_\gamma(\omega) + | G_2(\omega) |^2 R_\gamma(0) = S_\eta(\omega) + | G_2(\omega) |^2 S_x(\omega), \tag{59}$$

where

$$S_v(\omega) = \sum_{m=-\infty}^{\infty} R_v(m) \exp(-jm\omega) \tag{60}$$

is tbe power spectral density of v.

The variance of γ is calculated as

$$\sigma_\gamma^2 = R_\gamma(0) = \frac{1}{\pi}\int_0^\infty S_\gamma(\omega)\, d\omega \tag{61}$$

$$\sigma_\gamma^2 = \frac{\dfrac{1}{\pi}\displaystyle\int_0^\infty \left| \frac{G_2(\omega)}{1 + G_2(\omega)} \right|^2 S_x(\omega)\, d\omega + \dfrac{1}{\pi}\displaystyle\int_0^\infty \frac{S_\eta(\omega)}{| 1 + G_2(\omega) |^2}\, d\omega}{1 + \dfrac{1}{\pi}\displaystyle\int_0^\infty \left| \frac{G_2(\omega)}{1 + G_2(\omega)} \right|^2 d\omega}. \tag{62}$$

Since $G_2(\omega)$ is narrowband, we may assume tbat

$$\int_0^\infty \left| \frac{G_2(\omega)}{1 + G_2(\omega)} \right|^2 d\omega \ll 1 \tag{63}$$

and therefore,

$$\sigma_\gamma^2 \approx \frac{1}{\pi}\int_0^\infty \left| \frac{G_2(\omega)}{1 + G_2(\omega)} \right|^2 S_x(\omega)\, d\omega + \frac{1}{\pi}\int_0^\infty \frac{S_\eta(\omega)}{| 1 + G_2(\omega) |^2}\, d\omega. \tag{64}$$

The second term indicates that low-frequency components of tbe reference source noise are attenuated while higb-frequency components are not. Tberefore, if the reference source is derived from transmitted pilot tones, it should be filtered to a narrow bandwidth before being used in the timing recovery system.

In Appendix B, $S_x(\omega)$ is evaluated for $|\omega| \ll 1$. This is the only region of interest when $G_2(\omega)$ is a sufficiently narrow low-pass filter.

$$S_x(\omega) = \bar{e}_2^2 + \frac{2}{b^2}[R_n(0) - R_n(1)] + A_1 + A_2\omega^2, \tag{65}$$

where

$$A_1 = \frac{1}{b^2}\sum_{k=-\infty}^{\infty}[f(k+1/2) - f(k-1/2)] \cdot [2f(k+1/2) + f(-k+1/2) - f(-k-1/2)] \tag{66}$$

and

$$A_2 = \frac{1}{2b^2}\left\{-4f^2(1/2) + 2\sum_{k=-\infty}^{\infty} f(k+1/2)f(k-1/2) - \sum_{k=-\infty}^{\infty} k^2[f(k+1/2) - f(k-1/2)][f(-k+1/2) - f(-k-1/2)]\right\}. \tag{67}$$

If we let

$$\omega_1 = \frac{1}{\pi}\int_0^\infty \left|\frac{G_2(\omega)}{1+G_2(\omega)}\right|^2 d\omega \tag{68}$$

and

$$\omega_2^3 = \frac{1}{\pi}\int_0^\infty \omega^2 \left|\frac{G_2(\omega)}{1+G_2(\omega)}\right|^2 d\omega \tag{69}$$

then (64) becomes

$$\sigma_\gamma^2 = \bar{e}_2^2\omega_1 + \frac{2}{b^2}[R_n(0) - R_n(1)]\omega_1 + A_1\omega_1 + A_2\omega_2^3 + \frac{1}{\pi}\int_0^\infty \frac{S_\eta(\omega)}{|1+G_2(\omega)|^2}\, d\omega. \tag{70}$$

This equation is given in the summary with the normalization $T = 1$ removed. An application to a typical data transmission system is given in Section IV.

It should be noted from (69) that ω_2 will he unhounded unless $G_2(\omega)$ has at least two more poles than zeros. Good design of $G_2(\omega)$ requires that the second pole (assuming no zeros) occur somewhere in the vicinity of gain crossover. In this case, ω_2 is approximately equal to ω_1.

The first term of (70) indicates that the standard deviation of the jitter caused by frequency offset will he much less than the mean

of that error. The second term is the jitter produced by the additive noise. In typical data transmission systems, the signal is filtered such that $R_n(1) \approx 0$. In that case, the variance of the jitter is directly proportional to the noise power. Of particular interest is the jitter produced by signal overlap. It can be seen from (66) that $A_1 = 0$ if $f(t)$ is an even function. In that case, the jitter variance is proportional to the cube of the system bandwidth, and can be made quite small by the use of narrow filtering. If $f(t)$ is not an even function, then the third term will usually be much larger than the fourth term. In either case, the jitter will greatly increase as the filter used to shape $f(t)$ is made sharper.

III. THE SAMPLED-DERIVATIVE DETECTOR

An alternative method of adjusting the phase of the sampling wave makes use of the time derivative of the signal at the sampling times. Implementation of a sampled-derivative timing recovery system is about equally complex as a threshold crossing system.

Except for the manner in which the error signals are generated, the control loop is the same for both systems. Fig. 1 may be used to describe the sampled-derivative system if the delay in the feedback path is eliminated and a sampled-derivative detector is substituted for the threshold crossing detector.

The sampled-derivative detector generates an error pulse during each bit interval whose amplitude is proportional to the time derivative of the data signal at the sampling time, multiplied by the polarity of the signal at that time

$$e(t) = K_3 \sum_n \operatorname{sgn} [s(nT + \gamma T)] s'(nT + \gamma T)\ \delta(t - nT). \tag{71}$$

However, the output data is generated by setting

$$\hat{a}_n = \operatorname{sgn} [s(nT + \gamma T)], \tag{72}$$

where $\hat{a}_n$ is the receiver decision on the nth bit. If the error rate is low, $\hat{a}_n = a_n$ with high probability, and (71) may be approximated by

$$e(t) \approx K_3 \sum_n a_n s'(nT + \gamma T)\ \delta(t - nT) \tag{73}$$

if the effect of errors is neglected.

Using (1),

$$e(t) = K_3 \sum_n a_n [\sum_k a_{n-k} f'(kT + \gamma T - \beta T) + n'(t)]\ \delta(t - nT). \tag{74}$$

The abscissa of $f(t)$ in this case is adjusted such that the origin coincides with the peak of $f(t)$.

$$f'(0) = 0. \tag{75}$$

If the phase error $\gamma - \beta$ is small, we may approximate (74) by the first terms of its Taylor series expansion

$$e(t) \approx K_3 \sum_n a_n[a_n(\gamma - \beta)Tf''(0) + \sum_{k \neq 0} a_{n-k}f'(kT) + n'(t)]\ \delta(t - nT). \tag{76}$$

As in the previous case,

$$e(t) = \sum_n e(n)\ \delta(t - nT) \tag{77}$$

$$\gamma = K_2 \int_{-\infty}^{t} g_1(t - x)e(x)\ dx + \tau \tag{78}$$

and we normalize the time variable by setting $T = 1$. Equation (76) may now be written as

$$e(n) = K_3\Big[(\tau - \beta)f''(0) + a_n \sum_{k \neq 0} a_{n-k}f'(k) + a_n n'(t) + K_2 f''(0) \sum_{k=-\infty}^{n} g_1(n - k)e(k)\Big]. \tag{79}$$

Let

$$e_3(t) = -\frac{e(t)}{K_3 f''(0)} \tag{80}$$

and

$$g_3(t) = -K_2K_3f''(0)g_1(t). \tag{81}$$

Then (79) becomes

$$e_3(n) = \beta - \tau - y(n) - \sum_{k=-\infty}^{n} g_3(n - k)e_3(k), \tag{82}$$

where

$$y(n) = \frac{a_n}{f''(0)}\ [\sum_{k \neq 0} a_{n-k}f'(k) + n'(t)]. \tag{83}$$

Unlike the threshold crossing system, the sampled-derivative system is a time-invariant linear one when the phase error is small. A model of the system conforming with (82) is shown in Fig. 3. This model may be readily analyzed because of its linearity.

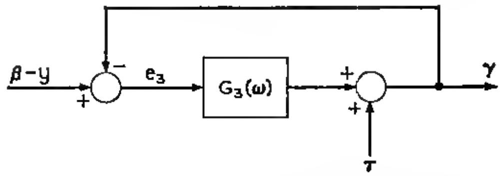

Fig. 3 — Model of sampled-derivative system.

$$\gamma(n) = \sum_{k=-\infty}^{n} g_3(n-k)e_3(k) + \tau \tag{84}$$

$$\gamma(n) = \sum_{k=-\infty}^{n} g_3(n-k)[\beta - y(k) - \gamma(k)] + \tau. \tag{85}$$

The mean error may be found from

$$\gamma(\omega)[1 + G_3(\omega)] = G_3(\omega)\beta(\omega) + \bar{\tau}(\omega) \tag{86}$$

$$\gamma(\omega) - \beta(\omega) = -E_3(\omega) = \frac{\bar{\tau}(\omega) - \beta(\omega)}{1 + G_3(\omega)}. \tag{87}$$

Equation (87) is identical to (46) if $G_3(\omega)$ is substituted for $G_2(\omega)/[1 + g_2(0)]$. All the comments of Section II concerning the static error and the desirability of $G(\omega)$ having a pole at the origin therefore, also apply to the sampled-derivative system.

The variance of γ when the mean error is constant will now be found.

$$\gamma_1(n) = \gamma(n) - \bar{\gamma} = -\sum_{k=-\infty}^{\infty} g_3(n-k)[y(k) + \gamma_1(k)] + \eta(t), \tag{88}$$

where $\eta(t)$ is again the reference source jitter. In terms of the power spectral densities,

$$S_\gamma(\omega) = \left| \frac{G_3(\omega)}{1 + G_3(\omega)} \right|^2 S_y(\omega) + \frac{S_\eta(\omega)}{| 1 + G_3(\omega) |^2}. \tag{89}$$

In Appendix C it is shown that, for $| \omega | \ll 1$,

$$S_y(\omega) \approx -\frac{R_n''(0)}{f''^2(0)} + A_3 + A_4\omega^2, \tag{90}$$

where

$$A_3 = \frac{1}{f''^2(0)} \sum_{k=-\infty}^{\infty} f'(k)[f'(k) + f'(-k)] \tag{91}$$

and

$$A_4 = -\frac{1}{2f''^2(0)} \sum_{k=-\infty}^{\infty} k^2 f'(k) f'(-k). \tag{92}$$

The variance of γ is then

$$\sigma_\gamma^2 = \frac{1}{\pi} \int_0^\infty S_\gamma(\omega)\, d\omega \tag{93}$$

$$\sigma_\gamma^2 = -\frac{R_n''(0)}{f''^2(0)} \omega_1 + A_3 \omega_1 + A_4 \omega_2^3 + \frac{1}{\pi} \int_0^\infty \frac{S_\eta(\omega)}{|1 + G_3(\omega)|^2}\, d\omega, \tag{94}$$

where

$$\omega_1 = \frac{1}{\pi} \int_0^\infty \left| \frac{G_3(\omega)}{1 + G_3(\omega)} \right|^2 d\omega \tag{95}$$

and

$$\omega_2^3 = \frac{1}{\pi} \int_0^\infty \omega^2 \left| \frac{G_3(\omega)}{1 + G_3(\omega)} \right|^2 d\omega. \tag{96}$$

There is a very strong similarity between (94) and (70). The last terms are identical, so that it is just as important in the sampled-derivative system as in the threshold crossing system that high-frequency noise components be removed from the referenced source prior to use for timing recovery.

The jitter due to additive noise is proportional to the power of the derivative of the noise. The example in the next section illustrates that this is not serious if the noise is bandlimited to the same frequency range as the signal. However, if any high-frequency noise is allowed to enter the receiver beyond the signal filter, the jitter will be greatly increased.

The jitter due to signal overlap is very similar to that of the threshold crossing system. If $f(t)$ is an even function, then its derivative will be odd, and $A_3 = 0$. Both A_3 and A_4 increase markedly as the spectrum of $f(t)$ is made sharper.

Finally, unlike the threshold crossing system, there is no additional jitter term due to static phase error. This jitter component is eliminated because there is an error pulse generated during each bit interval.

IV. AN EXAMPLE

In order to illustrate the results of the previous two sections, the output jitter of both a threshold crossing timing recovery system and

a sampled-derivative system will be calculated for a typical received data transmission signal.

The signal to be considered is one using half raised-cosine amplitude shaping which is distorted by linear delay distortion. For simplicity, it is normalized so that $T = 1$ and the undistorted peak of the signal is 1.

$$F(\omega) = A(\omega) \exp [j\varphi(\omega)] \tag{97}$$

$$A(\omega) = \begin{cases} 1, & 0 < \omega < \dfrac{\pi}{2} \\ \cos^2 \dfrac{\omega - \frac{\pi}{2}}{2}, & \dfrac{\pi}{2} < \omega < \dfrac{3\pi}{2} \end{cases} \tag{98}$$

$$\varphi(\omega) = \frac{3}{4\pi}\omega^2, \qquad \omega > 0. \tag{99}$$

The outline of the "eye pattern" for this signal is shown in Fig. 4, along with the central portion of $f(t)$. The eye pattern is formed by superimposing the signals of all possible message sequences. Closing of the eye is due to signal overlap.

The time of maximum eye opening is the optimum sampling time in a minimax sense. When such a sampling instant is chosen, then the error

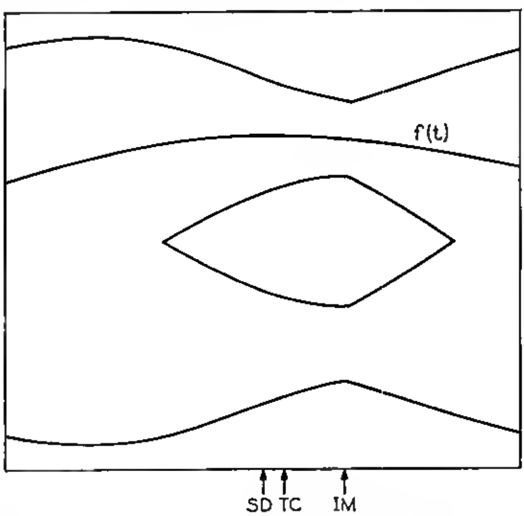

Fig. 4 — Eye pattern outline and mean sampling times for a typical distorted data signal.

probability in the presence of additive noise is minimized for the most adverse message sequence, where all adjacent bits overlap in a manner that subtracts from the bit being detected.

The mean sampling times are also shown in Fig. 4 for the case of no static error. The mean sampling time of threshold crossing system, TC, is such that $f(TC + \frac{1}{2}) = f(TC - \frac{1}{2})$. The mean sampling time of the sampled-derivative system, SD, coincides with the peak of $f(t)$. It is seen that the threshold crossing system chooses a better mean sampling time than the sampled-derivative system, yet both systems miss the maximum of the eye opening by a large amount.

In order to compare the jitter due to noise, a particular noise spectrum must be considered. Here it will be assumed that the noise is white noise which has been passed through a receive filter matched to the undistorted signal. In this case, the noise power spectral density is of the form

$$N(\omega) = \frac{\sigma_n^2 A(\omega)}{\frac{1}{\pi}\int_0^\infty A(\omega)\, d\omega} \tag{100}$$

so that

$$R_n(0) = \sigma_n^2 , \tag{101}$$

$$R_n(1) = 0, \tag{102}$$

and

$$R_n''(0) = -\frac{\int_0^\infty \omega^2 A(\omega)\, d\omega}{\int_0^\infty A(\omega)\, d\omega}\, \sigma_n^2 . \tag{103}$$

From (70), the rms jitter due to noise for the threshold crossing system is calculated to be

$$\sigma_\gamma = 0.615\sigma_n\sqrt{\omega_1} . \tag{104}$$

For the sampled-derivative system, this quantity is calculated from (94).

$$\sigma_\gamma = 0.612\sigma_n\sqrt{\omega_1} . \tag{105}$$

The results are virtually identical. In either case, if $\sigma_n = 0.1$ (signal-to-noise ratio of 20 dB) and $\omega_1 = 0.01$, then the rms jitter due to noise alone will be 0.61 percent. It should be mentioned that several other signal pulse shapes were examined, and it was found that the jitter due to noise was not very sensitive to pulse shape.

In order to calculate the jitter due to signal overlap, A_1 and A_3 were computed from (66) and (91). A_2 and A_4 were small enough to have negligible effect on the jitter. The resultant rms jitter is $0.287\sqrt{\omega_1}$ for the threshold crossing system and $0.265\sqrt{\omega_1}$ for the sampled-derivative system. If $\omega_1 = 0.01$, the jitter is 2.87 percent and 2.65 percent, respectively. This is by no means negligible, and illustrates the need for very narrow filtering in the timing recovery control loop. Again there is little difference in the performance of the two systems.

To observe the effects of asymmetry of the signal pulse, let us consider the same signal without phase distortion. Both timing recovery systems will then set a mean sampling time at the best point. The jitter due to signal overlap is greatly reduced since A_1 and A_3 are zero. The computed values of A_2 and A_4 are 0.11 and 0.32, respectively. If $\omega_3 = 0.01$, then the rms jitter due to signal overlap is only 0.01 percent for the threshold crossing detector and 0.03 percent for the sampled-derivative detector. Both values are completely negligible. It may be concluded from this calculation that both timing recovery systems are very sensitive to asymmetry of the signal waveform, both in terms of choosing the average sampling time and the resultant jitter about that time.

V. AN IMPROVED TIMING RECOVERY SYSTEM

It was seen in the previous example that both the threshold crossing timing recovery system and the sampled-derivative system led to average sampling times which differed considerably from the time of maximum eye opening. However, it is possible to modify the sampled-derivative system so that it does seek the time of maximum eye opening as the average sampling time.

At any time t_0, the signal amplitude for the worst message sequence, assuming the current bit is 1, is

$$D(t_0) = f(t_0) - \sum_{k \neq 0} | f(t_0 + kT) | . \tag{106}$$

In the region where the eye is open, $D(t_0) > 0$, and the eye opening is equal to $2D(t_0)$. If a sampling time t_0 is chosen such that $D(t_0) < 0$, then errors will occur for some sequences even in the absence of noise.

An experimental examination of the eye patterns of a large number of actual data transmission systems indicates that $D(t_0)$ is almost always a concave function of t_0. Therefore, if t_0 is adjusted according to the gradient of $D(t_0)$, then the maximum of D will be found.

It is therefore desired to generate an error signal whose average

value is

$$\overline{e(t)} = KD'(t - \beta T + \gamma T). \tag{107}$$

If we again normalize the system so that $T = 1$,

$$\overline{e(t)} = K[f'(t - \beta + \gamma) - \sum_{k \neq 0} f'(t + k - \beta + \gamma) \operatorname{sgn} f(t + k - \beta + \gamma)]. \tag{108}$$

Equation (108) exists and is continuous except at those points t_k where $f(t_k + k - \beta + \gamma) = 0$.

Fig. 5 is a block diagram of a system which generates an error signal whose average is given by (108). The first term of (108) is the average error signal of the sampled-derivative detector discussed in Section III. The improved timing recovery system therefore, will consist of a sampled-derivative system with added correction signals. Enough correction terms are used to account for those adjacent bits which may be expected to overlap significantly into the bit interval under consideration.

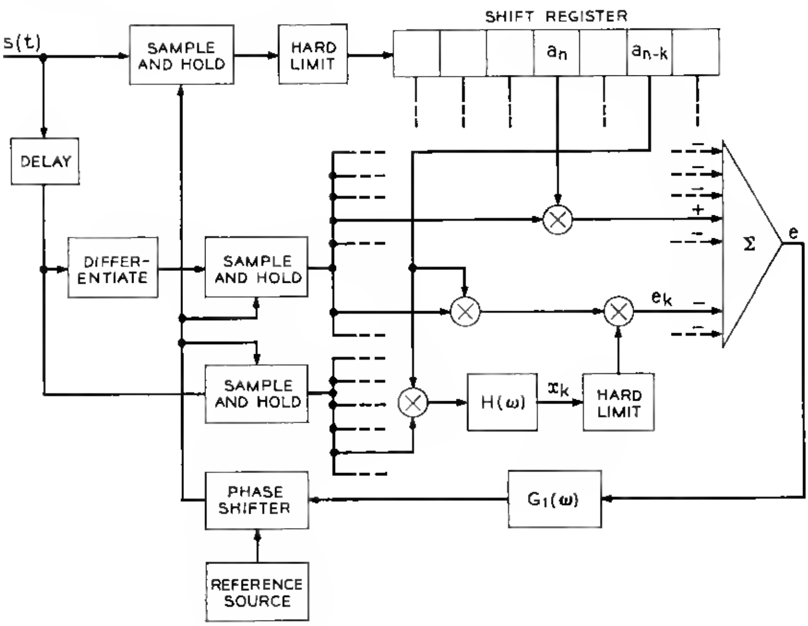

Fig. 5 — Improved timing recovery system.

The $k$tb correction signal must bave an average value

$$\overline{e_k(t_1)} = Kf'(k + t_1) \operatorname{sgn} f(k + t_1), \tag{109}$$

where

$$t_1 = t - \beta + \gamma. \tag{110}$$

This correction signal is formed by first generating an auxiliary signal $x_k(n)$ whose polarity is expected to agree witb that of $f(k + t_1)$. The derivative of tbe signal at the current sampling time is multiplied by tbe polarity of the signal at a time displaced from the current time by k bit intervals. The result is multiplied by the polarity of tbe auxiliary signal to form the correction signal.

$$e_k(n + t_1) = Ks'(n + t_1) \operatorname{sgn} s(n - k + t_1) \operatorname{sgn} x_k(n + t_1). \tag{111}$$

In order to account for overlap into leading pulses as well as lagging pulses, a fixed delay must be built into the system, as indicated in Fig. 5. This delay is equal to balf the shift register length, so tbat tbe central cell of the shift register stores the polarity of tbe current bit, wbile tbe other cells store tbe polarities of preceding and succeeding bits.

The auxiliary signal is formed by multiplying tbe value of tbe signal at the current sampling time by tbe polarity of tbe signal which preceded this signal by k bit intervals. If k is negative, tbe polarity of a succeeding bit is used. The resultant is filtered by a narrow filter, $H(\omega)$, to form tbe auxiliary signal x_k .

$$x_k(n) = \sum_m h(n - m)s(m) \operatorname{sgn} s(m - k), \tag{112}$$

wbere tbe time displacement t_1 is ignored.

If we assume tbat the error rate is low, as was done in Section III, then we may approximate $\operatorname{sgn} s(n) = a_n$ with little loss of accuracy. Using tbis approximation and substituting (1) into (112),

$$x_k(n) = \sum_m h(n - m)[a_{m-k}n(t) + f(k) + a_{m-k} \sum_{p \neq k} a_{m-p}f(p)]. \tag{113}$$

The mean of x_k is

$$\overline{x_k} = f(k)H(0). \tag{114}$$

In Appendix D it is shown that tbe variance of x_k is

$$\sigma_{x_k}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} | H(\omega) |^2 \left[\sigma_n^2 + \frac{1}{2\pi} \int_{-\infty}^{\infty} | F(u) |^2 \, du + P_k(\omega) \right] d\omega, \tag{115}$$

wbere $P_k(\omega)$ is tbe Fourier transform of

$$p_k(t) = f(k - t)f(k + t). \tag{116}$$

If $H(\omega)$ is sufficiently narrowband, then it can be seen from (115) that the variance of x_k will be small. Then, if $f(k)$ is not too small, it may be expected that the polarity of x_k will agree with that of $f(k)$ with high probability.

We will now examine the mean value of the correction signal. Using the assumption of low error probability and substituting (1) into (111), the correction signal is

$$e_k(n + t_1) = Ka_{n-k} \sum_l a_{n-l}f'(n + t_1) \operatorname{sgn} x_k(n + t_1). \tag{117}$$

In order to find the mean of (117), we make the approximation that $x_k(n)$ is independent of a_{n-k} and a_{n-l}. As a justification of this approximation, note from (113) that

$$\overline{x_k} \mid a_{n-k}, a_{n-l} = f(k)H(0) + f(l)h(0) + f(2k - l)h(l - k) \tag{118}$$

$$\overline{x_k} \mid a_{n-k}, a_{n-l} \approx \overline{x_k} \tag{119}$$

since $h(n) \ll H(0)$ for a narrowband filter. Let

$$P_k = \text{Prob}\,[\operatorname{sgn} x_k = \operatorname{sgn} f(k)]. \tag{120}$$

Then

$$\overline{e_k} \approx K(2P - 1)f'(k + t_1) \operatorname{sgn} f(k + t_1). \tag{121}$$

When the magnitude of $f(k + t_1)$ is sufficiently large, $P \approx 1$ and the mean of the kth correction signal is approximately the desired value. In the vicinity of a zero of $f(k + t_1)$, $\frac{1}{2} < P < 1$, and the correction signal will at least have the correct polarity, although not the correct magnitude. Ideally, the correction term should be a discontinuous function of t_1 at a zero of $f(k + t_1)$. The actual correction signal will have a mean value which is continuous, but the sharpness of change in the vicinity of a zero will increase as $H(\omega)$ is made narrower.

The rms jitter of this timing recovery system is extremely difficult to evaluate because of the presence of many nonlinear operations. However, this jitter may be expected to be much greater than that of a sampled-derivative system, since each correction signal may be expected to introduce jitter of the same order of magnitude as the main error signal. Narrow filtering in the control loop is therefore essential.

The mean sampling for the example of Section IV when this improved timing recovery system is used is shown in Fig. 4 as "IM". It is seen that the time of maximum eye opening has been found. For this example, only one leading and one lagging correction term were sufficient to choose this mean sampling time.

VI. EXTENSIONS

Each of the timing recovery systems described here may be modified to work with m-level digital data signals instead of only binary formats. Since in multilevel systems the eye is much narrower, the effects of static timing error and jitter are much more serious.

A threshold crossing detector may be constructed which generates an error pulse whenever the signal crosses any one of the $m - 1$ thresholds. During any signal interval, any number of error pulses between zero and $m - 1$ may be present. Such a system is extremely difficult to analyze, but has been found to work well in practice, provided that some auxiliary means is used to correct the mean sampling time.

The sampled-derivative detector is very easily extended to multilevel systems if the signal derivative is multiplied by the output symbol value in forming the error signal. If the a_n's are scaled so as to form a set of unit variance, then the analysis of Section III applies directly.

The improved timing recovery system is modified for use with multilevel signaling in a manner similar to that of the sampled-derivative system. However, the signal margin against noise for the worst message sequence is now

$$D(t_0) = f(t_0) - (m - 1) \sum_{k \neq 0} | f(t_0 + kT) |. \tag{122}$$

Comparing this criterion with that of (106), it is seen that each of the correction signals must be weighted by the quantity $m - 1$ in order to find the time of maximum eye opening.

Extension of these techniques to partial response systems is also straightforward. Since the modifications depend on the particular partial response system used, a description will not be presented here.

All of the systems analyzed here used linear control loops. This permitted the calculation of jitter variance in terms of loop bandwidth. However, the implementation of these systems may frequently be simplified considerably by using nonlinear control systems. A particular method which has met with practical success uses the polarity of each error pulse to adjust the sampling phase by a small fixed increment.

APPENDIX A

*Evaluation of $\overline{e_2}$**

Equation (40) is of the form

$$e_2(n) = 2d_n\left[c_n - \sum_{k=-\infty}^{n} g_2(n - k)e_2(k)\right], \tag{123}$$

* The approach used here was suggested by J. E. Mazo.

where $d_n = 0$ or 1 with equal probability and are independent. The random variables c_n are uncorrelated with the d_n's.

From (123) and the properties of d_n,

$$e_2(n) = d_n e_2(n) \tag{124}$$

and

$$\overline{d_n e_2(k)} = \tfrac{1}{2}\overline{e_2(k)}, \quad \text{if} \quad n > k. \tag{125}$$

We first find the average of (123) over all c_n and all d_k, $k < n$. This average of a random variable v will be denoted by $\langle v \rangle$, while the average over all c_n and d_n will be shown as $\bar{v}$.

$$\langle e_2(n) \rangle = 2d_n\bar{c}_n - 2\sum_{k=-\infty}^{n} g_2(n-k)\langle d_n e_2(k) \rangle. \tag{126}$$

Using (124)

$$\langle e_2(n) \rangle = 2d_n\bar{c}_n - 2g_2(0)\langle e_2(n) \rangle - 2\sum_{k=-\infty}^{n-1} g_2(n-k)\langle d_n e_2(k) \rangle. \tag{127}$$

The only random variable in (127) is d_n. The overall average, $\overline{e_2(n)}$, is therefore the average of (127) over d_n. Using (125), we obtain

$$\overline{e_2(n)} = \bar{c}_n - 2g_2(0)\overline{e_2(n)} - \sum_{k=-\infty}^{n-1} g_2(n-k)\overline{e_2(k)} \tag{128}$$

$$\overline{e_2(n)} = \bar{c}_n - g_2(0)\overline{e_2(n)} - \sum_{k=-\infty}^{n} g_2(n-k)\overline{e_2(k)} \tag{129}$$

which is the result shown in (41).

APPENDIX B

Evaluation of $S_x(\omega)$

From the definition of d_n in (29), we may express $2d_n$ as

$$2d_n = 1 - a_n a_{n+1}. \tag{130}$$

Then (51) may be rewritten as

$$x(n) \approx -a_n a_{n+1}\vec{e}_2 + z(n) + \frac{1}{b}(a_n - a_{n+1})n(t), \tag{131}$$

where the term $g_2(0)\bar{e}_2$ has been neglected.

We wish to evaluate the power spectral density of x. It will first be shown that the approximation of x given in (131) is zero-mean. Since the a_n's are zero-mean and independent, and the noise $n(t)$ is zero-

mean and independent of all other random processes, then the first and last terms of (131) are zero-mean. The mean of z is readily found after substituting (130) into (50).

$$z(n) = \frac{1}{b}(a_n - a_{n+1}) \sum_{k \neq 0, -1} a_{n-k} f(k + \tfrac{1}{2}) \tag{132}$$

$$\overline{z(n)} = 0. \tag{133}$$

The mean value of $x(n)$ as given in (131) is therefore zero. It may similarly be shown that the three terms of (131) are mutually uncorrelated. The autocorrelation of x is then

$$R_x(m) = \overline{a_n a_{n+1} a_{n+m} a_{n+m+1}}\, \overline{e_2^2} + R_z(m) + \frac{1}{b^2} \overline{(a_n - a_{n+1})(a_{n+m} - a_{n+m+1})} R_n(m) \tag{134}$$

$$R_x(0) = \overline{e_2^2} + R_z(0) + \frac{2}{b^2} R_n(0) \tag{135}$$

$$R_x(\pm 1) = R_z(\pm 1) - \frac{1}{b^2} R_n(\pm 1) \tag{136}$$

$$R_x(m) = R_z(m), \qquad m \neq 0, \quad \neq 1. \tag{137}$$

From (132),

$$R_z(m) = \frac{1}{b^2} \sum_{k \neq 0, -1} \sum_{l \neq 0, -1} \overline{(a_n - a_{n+1})(a_{n+m} - a_{n+m+1}) a_{n-k} a_{n+m-l}} \cdot f(k + \tfrac{1}{2}) f(l + \tfrac{1}{2}) \tag{138}$$

$$R_z(0) = \frac{2}{b^2} \sum_{k \neq 0, -1} f^2(k + \tfrac{1}{2}) \tag{139}$$

$$R_z(\pm 1) = -\frac{1}{b^2} [\sum_{k \neq 0, \pm 1} f(k + \tfrac{1}{2}) f(k - \tfrac{1}{2}) + f(-\tfrac{3}{2}) f(\tfrac{3}{2})] \tag{140}$$

$$R_z(\pm m) = \frac{1}{b^2} [f(m + \tfrac{1}{2}) - f(m - \tfrac{1}{2})][f(-m + \tfrac{1}{2}) - f(-m - \tfrac{1}{2})], \qquad m \neq 0, \quad \neq 1. \tag{141}$$

The power spectral density of x can now be calculated for $|\omega| < \pi$.

$$S_x(\omega) = \sum_{m=-\infty}^{\infty} R_x(m) \exp(-j\omega m) \tag{142}$$

$$S_x(\omega) = R_x(0) + 2 \sum_{m=1}^{\infty} R_x(m) \cos \omega m. \tag{143}$$

Since x will be passed through a very narrow filter, we are interested in the spectrum of x only in the region $|\omega| \ll 1$. In this region, (143) may be approximated by

$$S_x(\omega) \approx R_x(0) + 2 \sum_{m=1}^{\infty} R_x(m) - \omega^2 \sum_{m=1}^{\infty} m^2 R_x(m). \tag{144}$$

This approximation is valid if the third derivative of $S_x(\omega)$ is bounded.[5] This will be true if $R_x(m)$ decreases as $O(m^{-5})$. If the Fourier transform of $f(t)$ is continuous, then $f(m)$ decreases as $O(m^{-2})$. In this case, it can be seen from (137) and (141) that $R_x(m)$ decreases as $O(m^{-6})$, so that the approximation (144) is valid for $|\omega| \ll 1$.

$$\begin{aligned} S_x(\omega) \approx \bar{e}_2^2 + \frac{2}{b^2} \Big\{ & R_n(0) - R_n(1) + \sum_{k \neq 0,-1} f^2(k + \tfrac{1}{2}) \\ & - \sum_{k \neq 0, \pm 1} f(k + \tfrac{1}{2}) f(k - \tfrac{1}{2}) - f(-\tfrac{3}{2}) f(\tfrac{3}{2}) \\ & + \sum_{m=2}^{\infty} [f(m + \tfrac{1}{2}) - f(m - \tfrac{1}{2})][f(-m + \tfrac{1}{2}) - f(-m - \tfrac{1}{2})] \Big\} \\ & + \omega^2 \Big\{ \sum_{k \neq 0, \pm 1} f(k + \tfrac{1}{2}) f(k - \tfrac{1}{2}) + f(-\tfrac{3}{2}) f(\tfrac{3}{2}) \\ & - \sum_{m=2}^{\infty} m^2 [f(m + \tfrac{1}{2}) - f(m - \tfrac{1}{2})][f(-m - \tfrac{1}{2}) - f(-m - \tfrac{1}{2})] \Big\}. \end{aligned} \tag{145}$$

After some manipulation, and using (26), (145) may be reduced to

$$\begin{aligned} S_x(\omega) = \bar{e}_2^2 + \frac{2}{b^2} \Big\{ & R_n(0) - R_n(1) + \frac{1}{2} \sum_{k=-\infty}^{\infty} [f(k + \tfrac{1}{2}) - f(k - \tfrac{1}{2})] \\ & \cdot [2f(k + \tfrac{1}{2}) + f(-k + \tfrac{1}{2}) - f(-k - \tfrac{1}{2})] \Big\} \\ & + \frac{\omega^2}{2b^2} \Big\{ -4f^2(\tfrac{1}{2}) + 2 \sum_{k=-\infty}^{\infty} f(k + \tfrac{1}{2}) f(k - \tfrac{1}{2}) \\ & - \sum_{k=-\infty}^{\infty} k^2 [f(k + \tfrac{1}{2}) - f(k - \tfrac{1}{2})][f(-k + \tfrac{1}{2}) - f(-k - \tfrac{1}{2})] \Big\}. \end{aligned} \tag{146}$$

APPENDIX C

Evaluation of $S_y(\omega)$

We wish to find the power spectral density of

$$y(n) = \frac{a_n}{f''(0)} \left[\sum_{k \neq 0} a_{n-k} f'(k) + n'(t) \right]. \tag{147}$$

The autocorrelation of y is

$$R_y(m) = \frac{1}{f''^2(0)} \Big[\sum_{k\neq 0} \sum_{l\neq 0} \overline{a_n a_{n+m} a_{n-k} a_{n+m-l}} f'(k) f'(l) + \overline{a_n a_{n+m} n'(t) n'(t+m)}\Big] \tag{148}$$

$$R_y(0) = \frac{1}{f''^2(0)} \Big[\sum_{k\neq 0} f'^2(k) - R_n''(0)\Big]. \tag{149}$$

For $m \neq 0$,

$$R_y(m) = \frac{1}{f''^2(0)} f'(m) f'(-m). \tag{150}$$

Under the same conditions stated in Appendix B, the power spectral density of y may be approximated in the region $|\omega| \ll 1$ by

$$S_y(\omega) \approx R_y(0) + 2 \sum_{m=1}^{\infty} R_y(m) - \omega^2 \sum_{m=1}^{\infty} m^2 R_y(m). \tag{151}$$

Using (75) we obtain

$$S_y(\omega) \approx \frac{1}{f''^2(0)} \Big\{-R_n''(0) + \sum_{k=-\infty}^{\infty} f'(k)[f'(k) + f'(-k)] - \tfrac{1}{2}\omega^2 \sum_{k=-\infty}^{\infty} k^2 f'(k) f'(-k)\Big\}. \tag{152}$$

APPENDIX D

Evaluation of σ_{xk}^2

The variance of x_k is the mean square value of the zero-mean random process

$$x_k(n) - \bar{x} = \sum_m h(n-m) a_{m-k} [n(t) + \sum_{m\neq p} a_{m-p} f(p)]. \tag{153}$$

Since the noise and the message are independent, the variances of the two components of (153) will add to form the total variance. The variance of x_k due to the noise is

$$\sigma_1^2 = \sum_m \sum_q h(n-m) h(n-q) \overline{a_{m-k} a_{q-k} n(t_m) n(t_q)} \tag{154}$$

$$\sigma_1^2 = \sum_m h^2(n-m) \overline{n^2(t)} \tag{155}$$

$$\sigma_1^2 = \frac{1}{2\pi} \sigma_n^2 \int_{-\infty}^{\infty} |H(\omega)|^2 \, d\omega. \tag{156}$$

The total variance of x_k is then

$$\sigma_{xk}^2 = \sigma_1^2 + \sum_m \sum_q \sum_{p \neq k} \sum_{r \neq k} h(n - m)h(n - q)\overline{a_{m-k}a_{q-k}a_{m-p}a_{q-r}} \cdot f(p)f(q). \tag{157}$$

Nonzero contributions arise from those terms where $q = m$ and $r = p$ and from those terms where $r = 2k - p$ and $q = m + k - p$.

$$\sigma_{xk}^2 = \sigma_1^2 + \sum_m \sum_{p \neq k} [h^2(n - m)f^2(p) + h(n - m)h(n - m - k + p)f(p)f(2k - p)]. \tag{158}$$

Any one of the terms of (158) is small compared to the sum. We may therefore approximate (158) by including the missing $p = k$ terms.

$$\sigma_{xk}^2 \approx \sigma_1^2 + \frac{1}{4\pi^2} \int_{-\infty}^{\infty} |H(\omega)|^2 \, d\omega \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega + \sigma_2^2 , \tag{159}$$

where

$$\sigma_2^2 = \sum_m \sum_p h(m)h(m - k + p)f(p)f(2k - p) \tag{160}$$

and it is assumed that $H(\omega)$ is bandlimited to $|\omega| < \pi$.

$$\sigma_2^2 = \frac{1}{16\pi^4} \iiiint_{-\infty}^{\infty} H(\omega)H(u)F(v)F(y) \sum_m \exp[jm(\omega + u)] \cdot \sum_p \exp[jp(u + v - y)] \exp[jk(2y - u)] \, d\omega \, du \, dv \, dy \tag{161}$$

$$\sigma_2^2 = \frac{1}{4\pi^2} \iiiint_{-\infty}^{\infty} H(\omega)H(u)F(v)F(y) \, \delta(\omega - u) \, \delta(u - v - y) \cdot \exp[jk(2y - u)] \, d\omega \, du \, dv \, dy \tag{162}$$

$$\sigma_2^2 = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} H(\omega)H(-\omega) \, d\omega \int_{-\infty}^{\infty} F(y - \omega)F(y) \exp[jk(2y - \omega)] \, dy \tag{163}$$

$$\sigma_2^2 = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} |H(\omega)|^2 \, d\omega \int_{-\infty}^{\infty} F^*(\omega - y) \exp[-jk(\omega - y)] \cdot F(y) \exp(jky) \, dy. \tag{164}$$

The second integral in (164) may be recognized as 2π times the Fourier transform of

$$p_k(t) = f(k - t)f(k + t) \tag{165}$$

so that

$$\sigma_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} | H(\omega) |^2 P_k(\omega) \, d\omega. \tag{166}$$

Substituting (166) and (156) into (159)

$$\sigma_{x_k}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} | H(\omega) |^2 \left[\sigma_n^2 + \frac{1}{2\pi} \int_{-\infty}^{\infty} | F(u) |^2 \, du + P_k(\omega) \right] d\omega. \tag{167}$$

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